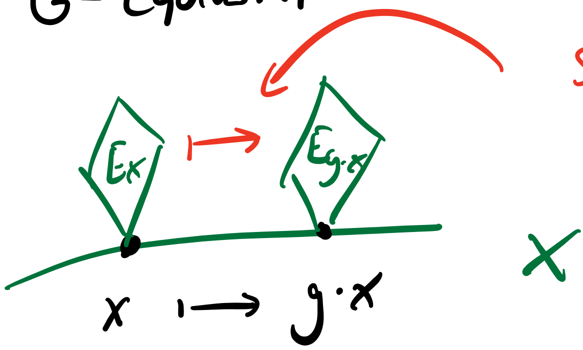
Resources: Oxford Lectures by Liu

$G \curvearrowright X$ giving $G \times X \to X$
group, scheme

e.g. a G-equivariant vector bundle E:



some linear map
$G \times E \to E$
lifting acts fiberwise

Ex) $GL(V) \curvearrowright \mathbb{P}(V)$, $\mathcal{O}(-1)$ has $GL(V)$-equivariant structure

$$g\cdot \underbrace{(x, v)}_{\in \mathcal{O}(-1)} = (\underbrace{g\cdot x}_{\in \mathbb{P}(V)}, g\cdot v)$$

$\Rightarrow$ its tensors & duals do too
i.e. $\mathcal{O}(n)$

$\mathcal{O}_X$-mod

Def: A G-equivariant structure on $\mathcal{F}$ is an iso

$$act^* \mathcal{F} \simeq p^* \mathcal{F}$$

$$G \times X \underset{p}{\overset{act}{\rightrightarrows}} X$$

E.g. $\mathcal{F}_{g\cdot x} \simeq \mathcal{F}_x$ is stalk at $(g, x) \in G \times X$

Compatible w/ group structure.

Ex) $X = pt$ sheaves $=$ vector spaces V — a rep of G

$$\underset{G}{\circlearrowleft}\quad\text{(sheaves)}\qquad \underset{G}{\circlearrowleft}\quad\text{(vector spaces)}$$

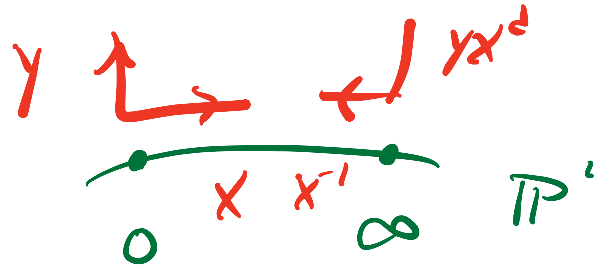
If $\dim V = 1$ $\qquad g\cdot v = \underbrace{\rho(g)}v$ $\qquad \rho : G \to \mathbb{C}^\times$

I think we're okay to say locally free here? vect. bundles / pt = vect. spaces

$\Rightarrow$ we say V has weight ρ

Ex) $X = \mathbb{P}^1$, line bundles are

$$\mathcal{L} = \mathcal{O}(d)$$



Let $\mathbb{C}^\times \curvearrowright \mathbb{P}^1$ by $t\cdot[x:y] = [\rho(t)x : y]$

If $\mathcal{L}$ is $\mathbb{C}^\times$-equivariant

$$\rho^{-d}\cdot \mathrm{wt}\, \mathcal{L}|_0 = \mathrm{wt}\, \mathcal{L}|_\infty \Big]??$$

Equivariant (algebraic) K-theory

<u>Def</u>:

$$K_G(X) = K(\mathrm{Coh}_G(X))$$

$=$ Grothendieck group of $\mathrm{Coh}_G(X)$

Generators: $\mathcal{F} \in \mathrm{Coh}_G(X)$

Relations: $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

Equivariant (topological) K-theory

$K_G^{vect}(X) =$ Grothendieck group of G-equiv vector bundles

i.e. locally free sheaves

$$\subset K_G(X)$$

<u>Ex</u> If X smooth

$$K_G(X) = K_G^{vect}(X)$$

Ex) $K(\mathbb{P}^1) = K^{vect}(\mathbb{P}^1) = \mathbb{Z}[\mathcal{L}^{\pm 1}]/\text{relns}$

$\mathcal{L} = \mathcal{O}(1)$

Ex) $X = pt$

$$K_G(pt) = Rep(G)$$

e.g. – $K_{\mathbb{C}^\times}(pt) = \mathbb{Z}[t^{\pm 1}]$

1-dim rep of weight 1

– $K_T(pt) = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]$

$T = (\mathbb{C}^\times)^n$

– $K_{GL(n)}(pt) = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]^{S_n}$

So, $K_G(pt) = \mathbb{Z}[\Lambda]$

Λ - character lattice

Functoriality: For $f: X \to Y$ to preserve exactness

$$f_*[\mathcal{G}] \neq [f_* \mathcal{G}]$$
$$= \sum_i (-1)^i [R^i f_* \mathcal{G}]$$

$$f^*[\mathcal{G}] \neq [f^* \mathcal{G}]$$
$$= \sum_i (-1)^i [L^i f^* \mathcal{G}]$$

Ex) $f: \mathbb{A}^1_{\mathbb{C}} \to pt$

$$\Rightarrow f_* \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}} = \mathbb{C}[x]$$

To compute, factor

$$f: X \hookrightarrow X \times Y \twoheadrightarrow Y$$

Ex)

$$0 \to \mathcal{O}_{\mathbb{C}^2} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}_{\mathbb{C}^2}^{\oplus 2} \xrightarrow{(x\ y)} \mathcal{O}_{\mathbb{C}^2} \to i_* \mathcal{O}_0 \to 0$$

wt t_1^{-1}

wt t_2^{-1}

embedding $\{0\} \hookrightarrow \mathbb{C}^2$

For $T = (\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2$ this is T-equivariant

$$i_*[\mathcal{O}_0] = 1 - (t_1 + t_2) + t_1 t_2$$
$$= (1 - t_1)(1 - t_2)$$
$$\in K_T(\mathbb{C}^2)$$

I don't understand

$$\iota_*[\mathcal{O}_0] = \sum_{i=0}^{2} (-1)^i [R^i \iota_* \mathcal{O}_0]$$

$$= [R^0 \iota_* \mathcal{O}_0] - [R^1 \iota_* \mathcal{O}_0] + [R^2 \iota_* \mathcal{O}_0]$$

Prototypical example: $\iota : X \hookrightarrow E$ zero section of a vector bundle

$$\cdots \to \pi^* \Lambda^2 E^\vee \to \pi^* E^\vee \to \mathcal{O}_E \to \iota_* \mathcal{O}_X \to 0$$

π is projection

e.g. $\pi : X \to pt$ has

$$\pi_*[\mathcal{F}] = \underbrace{\sum_k (-1)^k [H^k(X, \mathcal{F})]}_{\chi(X,\mathcal{F})}$$

Pullback:

$$f : X \xrightarrow{\iota} X \times Y \xrightarrow{\pi} Y$$

1) π is flat $\Rightarrow$ π^* is exact

$\Rightarrow$ $\pi^*[\mathcal{F}] = [\pi^* \mathcal{F}]$

2) ι closed embedding

$$\iota^* \mathcal{F} := \mathcal{O}_X \otimes_{\iota^{-1}\mathcal{O}_Y} \iota^{-1}\mathcal{F}$$

$$\iota^*[\mathcal{F}] = \sum_k (-1)^k \left[\mathrm{Tor}_k^Y(\iota_*\mathcal{O}_X, \mathcal{F})\right]$$

Tensor product